Investigating Goodstein Sequences

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September 12, 2025

Abstract

The weak and strong Goodstein theorems are examples of strongly counter intuitive results concerning certain integer sequences that typically grow very rapidly but eventually converge to zero. In this paper we describe the weak Goodstein sequences and aim provide the reader with an *intuitive* understanding of why the sequences converge. After that, although the paper remains descriptive, it takes a mathematical turn. We introduce transfinite ordinal numbers and demonstrate a decreasing sequence which bounds a weak Goodstein sequence above and terminates in zero. We consider the extraordinary behaviour of the strong Goodstein sequence and show how to construct corresponding decreasing sequences of transfinite ordinals. We call for help from an AI model to do the heavy lifting we require for our algebraic manipulations.

1 Introduction

A weak Goodstein sequence is constructed by choosing any number and and expressing it in base 2. Successive numbers are formed by reinterpreting the expression of the current number after incrementing the base, then subtracting 1. The Forth program G performs these steps, taking as input the first number in the current sequence. Starting with 266 the first two steps are:

$266\ G$
$100001010_2 \ = \ 266_{10}$
$100001010_3 - 1 = 100001002_3 = 6590_{10}$

At each step we increase the base, in this case from 2 to 3. Reinterpreting the same string of characters in the new base gives a value which has increased from 266 to 6591. We then subtract 1.

Here are the following three steps:

$100001002_4 - 1 = 100001001_4 = 65601_{10}$
$100001001_5 - 1 = 100001000_5 = 390750_{10}$
$100001000_6 - 1 = 100000555_6 = 1679831_{10}$

Increasing the base and reinterpreting the same string of figures has an enormous effect, again which the effect of subtracting 1 seems relatively insignificant. So why do such sequences converge to zero?

One initial hint can be seen in the above example. Although successive terms grow rapidly in the above example their representations do not grow in size.

2 A small example

We take the example if $4 = 100_2$.

Reinterpreting the string 100 in base 3 gives us $100_3 = 9_{10}$.

Now subtracting 1 we see $100_3 - 1 = 22_3$ Subtracting 1 has required a carry and in this case has reduced the size of the representation from 3 figures to 2.

Let's follow the process and see if the 2 figures will eventually reduce to 1.

4 G
$100_2 = 4_{10}$
$100_3 - 1 = 22_3 = 8_{10}$
$22_4 - 1 = 21_4 = 9_{10}$
$21_5 - 1 = 20_5 = 10_{10}$
$20_6 - 1 = 15_6 = 11_{10}$
$15_7 - 1 = 14_7 = 11_{10}$
$14_8 - 1 = 13_8 = 11_{10}$
$13_9 - 1 = 12_9 = 11_{10}$
$12_{10} - 1 = 11_{10} = 11_{10}$
$11_{11} - 1 = 10_{11} = 11_{10}$
$10_{12} - 1 = B_{12} = 11_{10}$

When we arrive at base 6 our representation has a leading 1. At this point increasing the base increments the sequence value by 1 and subtracting 1 reduced it by 1 so the sequence values are identical until we arrive at base 12.

At this point our representation is $10_{12} - 1 = B_{12}$ and we have reduced our representation to a single figure. With a single figure representation increasing the base has no effect, so our sequence terms decrease by 1 at each step as follows:

$10_{12} - 1 = B_{12} = 11_{10}$
$B_{13} - 1 = A_{13} = 10_{10}$
$A_{14} - 1 = 9_{14} = 9_{10}$
$9_{15} - 1 = 8_{15} = 8_{10}$
$8_{16} - 1 = 7_{16} = 7_{10}$
$7_{17} - 1 = 6_{17} = 6_{10}$
$6_{18} - 1 = 5_{18} = 5_{10}$
$5_{19} - 1 = 4_{19} = 4_{10}$
$4_{20} - 1 = 3_{20} = 3_{10}$
$3_{21} - 1 = 2_{21} = 2_{10}$
$2_{22} - 1 = 1_{22} = 1_{10}$
$1_{23} - 1 = 0_{23} = 0_{10}$

3 Potential and Achievement

Following the above discussion we wonder if we might be able to introduce some definitions that that in some sense can capture the eventual reduction of the length of our representation is a more finely calibrated way. With this in mind we introduce Potential and Achievement.

Returning to our first example of $266 = 100001010_2$ we are starting with a number which requires 9 bits for its representation. The maximum value we can represent in binary with 9 characters at our disposal is 512-1 = 511. We call this the potential of a 9 bit binary number. The value 100001010_2 achieves $266/511 = 0.5205$ of its potential, and we will say it has an achievement of .5205. The maximum value we can represent in 9 places with a base 3 representation is $3^9 - 1 = 19628$. The second value in our sequence is 6590, so its achievement is $6590/19628 = 0.3450$.

This pattern continues, so that as the values in the sequence initially increase before eventually decreasing, the achievements of the terms is always decreasing. Since an achievement of zero is associated with a number which is zero, if we could prove that the achievements converge to zero, this would prove the sequence converges to zero. However, proving the convergence of a sequence of real values terms to zero is not in general easy to do, wheras a decreasing sequence of positive whole numbers willalways decrease to zero. Could we perhaps produce a decreasing sequence of positive whole numbers to act as upper bounds to our sequence? With a sequence that increases to rapidly this would seem difficult, but we will do it by admitting transfinite numbers.

4 Transfinite Numbers - defining numbers with sets

The traditional proof of convergence for Goodstein sequences uses Cantors hierarchy of "ordinal numbers". We define numbers in terms of sets, as follows.

$0 \ \widehat{=} \ \{\ \}$ zero will be modelled by the empty set.
$1 \ \widehat{=} \ \{0\}$
$2 \ \widehat{=} \ \{0, 1\}$
...
$n \ \widehat{=} \ \{0, 1, 2, \ .. \ n-1\}$ n is defined as the set of all numbers *less than* n.

Then if numbers $a, b, \ a < b$ when $a \subseteq b$

The smallest set that is bigger than all the finite numbers is referred to as ω, and is defined as follows:

$\omega \ \widehat{=} \ \{0, 1, 2, \ ... \ \}$.

This is our first transfinite number. We can define its successor as follows:

$\omega + 1 \ \widehat{=} \ \{\omega, 0, 1, 2.....\}$

We can continue in this way defining a hierarchy of transfinite numbers

$\omega,\ \omega+1,\ \omega+2,\ ...\ 2\omega,\ 2\omega+1,\ ,,,\ 3\omega\ ,\ ...\ \omega^2,\ ...\ \omega^3,\ ...\omega^\omega,\ ....$

Note that although ω has a successor, it has no predecessor. $\omega - 1$ is undefined. It shares this property with other "limit ordinals", e.g. 2ω, ω^2, ω^ω etc.

5 Bounding the terms of a Goodstein sequence using ordinal numbers

The terms of any Goodstein sequence can be bounded above by a decreasing sequence of ordinals. A well known theorem states that any decreasing sequence of ordinals terminates in zero, and we can call on this theorem to show that any Goodstein sequence terminates in zero.

We give an example to show such a sequence of ordinals is created, and how it decreases.

Base	Term	Transfinite bound
2	$100_2 = 2^2$	ω^2
3	$3^2 - 1 = 2 \times 3 + 2$	$2\omega + 2$
4	$2 \times 4 + 2 - 1 = 2 \times 4 + 1$	$2\omega + 1$
5	$2 \times 5 + 1 - 1 = 2 \times 5$	2ω
6	$2 \times 6 - 1 = 6 + 5$	$\omega + 5$
7	$7 + 5 - 1 = 7 + 4$	$\omega + 4$
8	$8 + 4 - 1 = 8 + 3$	$\omega + 3$
9	$9 + 3 - 1 = 9 + 2$	$\omega + 2$
10	$10 + 2 - 1 = 10 + 1$	$\omega + 1$
11	$11 + 1 - 1 = 11$	ω
12	$12 - 1 = 11$	11
13	$11 - 1 = 10$	
...		

We have shown the sequence up to the point where the value of the current term is represented by a single figure. From this point onwards any increase in the base has no effect on the next value, so terms decrease by 1 at each step until 0 is reached.

6 The Strong Goodstein Sequence

Strong Goodstein sequences are specifically designed to grow very rapidly. The strong Goodstein sequence starting at 266 has the following values for its first 5 terms, yet eventually converges to zero.

$266,\ 4.4 \times 10^{36},\ 3.2 \times 10^{616},\ 2.5 \times 10^{10972}$

For perspective one can bear in mind that that the number of atoms in the observable universe is commonly estimated to be in the region of 10^{80}

A strong Goodstein sequence starts with a binary number expressed in "hereditary base notation". This notation restricts us to describing base n numbers using exponentiation on n and addition of numbers 1 to n-1 to fill in intermediate values. For example in base 2:

$$1 = 1,\ 2 = 2,\ 3 = 2+1,\ 4 = 2^2,\ 5 = 2^2+1,\ 6 = 2^2+2,$$
$$7 = 2^2+2+1,\ 8 = 2^{2+1},\ 16 = 2^{2^2}$$
$$31 = 2^{2^2} + 2^{2+1} + 2^2 + 2 + 1$$

The proof of convergence of the sequence constructs a corresponding decreasing sequence of transfinite terms obtained by replacing the base by ω. As an example we will take the sequence whose first term is 16, which is expressed in hereditary base notation as 2^{2^2} The initial transfinite term is obtained by replacing each 2 in the base 2 term with ω, giving ω^{ω^ω}. The next term in the numeric sequence is obtained by replacing each 2 in the hereditary base representation by 3 and subtracting 1, giving $3^{3^3} - 1$. However, this is not in hereditary base 3 form. The next transfinite term is given by expressing the numeric term in hereditary base 3 and replacing each occurrence of the base 3 by ω.

7 Help from an AI model

Converting $3^{3^3} - 1$ to hereditary base 3 and producing the related transfinite ordinal term requires non-standard and relatively heavy algebraic manipulation, so I was curious to see if it would be useful to enlist the help of an AI agent, in this case chatGPT 5, which had just been released and was supposed to be capable of PhD level maths. After initial incorrect attempts in which it did not interpreted hereditary base notation correctly, and I had to remind it that $\omega - 1$ was undefined, and after being given the hint that since $3^{3^{3-1}}$ has the form $x^3 - 1$, we can use the identity $x^3 - 1 = (x-1)(x^2+x+1)$, it did produce the complex formulae used in the rest of this section.

$$\begin{aligned}
&3^{3^3} - 1 = \\
&2\times 3^{2\times 3^2 + 2\times 3 + 2} + 2\times 3^{2\times 3^2 + 2\times 3 + 1} + 2\times 3^{2\times 3^2 + 2\times 3} + \\
&2\times 3^{2\times 3^2 + 3 + 2} + 2\times 3^{2\times 3^2 + 3 + 1} + 2\times 3^{2\times 3^2 + 3} + \\
&2\times 3^{2\times 3^2 + 2} + 2\times 3^{2\times 3^2 + 1} + 2\times 3^{2\times 3^2} + \\
&2\times 3^{3^2 + 2\times 3 + 2} + 2\times 3^{3^2 + 2\times 3 + 1} + 2\times 3^{3^2 + 2\times 3} + \\
&2\times 3^{3^2 + 3 + 2} + 2\times 3^{3^2 + 3 + 1} + 2\times 3^{3^2 + 3} + \\
&2\times 3^{3^2 + 2} + 2\times 3^{3^2 + 1} + 2\times 3^{3^2} + \\
&2\times 3^{2\times 3 + 2} + 2\times 3^{2\times 3 + 1} + 2\times 3^{2\times 3} + \\
&2\times 3^{3 + 2} + 2\times 3^{3 + 1} + 2\times 3^{3} + \\
&2\times 3^2 + 2\times 3 + 2
\end{aligned}$$

Note, we don't go directly to calculating the *value* of the expression. We keep it in this form, in which 3 is the current number base, so that we can obtain the corresponding transfinite term by replacing each 3 by ω.

$2\omega^{2\omega^2 + 2\omega + 2} + 2\omega^{2\omega^2 + 2\omega + 1} + 2\omega^{2\omega^2 + 2\omega} +$
$2\omega^{2\omega^2 + \omega + 2} + 2\omega^{2\omega^2 + \omega + 1} + 2\omega^{2\omega^2 + \omega} +$
$2\omega^{2\omega^2 + 2} + 2\omega^{2\omega^2 + 1} + 2\omega^{2\omega^2} +$
$2\omega^{\omega^2 + 2\omega + 2} + 2\omega^{\omega^2 + 2\omega + 1} + 2\omega^{\omega^2 + 2\omega} +$
$2\omega^{\omega^2 + \omega + 2} + 2\omega^{\omega^2 + \omega + 1} + 2\omega^{\omega^2 + \omega} +$
$2\omega^{\omega^2 + 2} + 2\omega^{\omega^2 + 1} + 2\omega^{\omega^2} +$
$2\omega^{2\omega + 2} + 2\omega^{2\omega + 1} + 2\omega^{2\omega} +$
$2\omega^{\omega + 2} + 2\omega^{\omega + 1} + 2\omega^{\omega} +$
$2\omega^2 + 2\omega + 2$

The important point about this expression is that it represents a transfinite number less than ω^{ω^ω}. We can see this by comparing ω^{ω^ω} to the highest order term in the expression.

A reader might conceivable wonder why go to such bother. Can't we just use $\omega^{\omega^\omega} - 1$ as our next transfinite term? The reason is that ω^{ω^ω} is a limit ordinal and has no predecessor, so $\omega^{\omega^\omega} - 1$ is undefined.

8 Background and Related Work

Reuben Goodstein introduced his sequences in the paper "On the restricted ordinal theorem", Journal of Symbolic Logic, Vol. 9, No. 2 (June 1944), pp. 33–41. The sequences were specifically designed to provide an application for transfinite ordinals, and hereditary base notation was specifically introduced for describing the strong sequences. In L. Kirby and J. Paris, "Accessible independence results for Peano Arithmetic," Bulletin of the London Mathematical Society 14 (1982), 285–293, we find a proof that convergence of the strong Goodstein sequence cannot be proved in the usual formalisation of arithmetic using Peano's axioms. This result is sometimes cited as an illustrative example for Gödel's incompleteness theorem, which states that all non-trivial mathematical theories must include results which are true but unprovable *within the theory.* An unpublished paper of Cansell and Abrial represents the strong Goodstein sequences as finite trees, and shows proof of their convergence can be deduced from the properties of these finite trees.

9 Conclusions

We've looked at Goodstein Sequences and tried to give some understanding of why these converge to zero despite initially growing so rapidly. The weak Goodstein sequence grows by interpreting the same string of figures in a new base, incremented by 1 at each step, then subtracts 1. Our first clue as to why the -1 steps wins out over the base increase is to notice that the number of figures used in the representation of each step does not increase. This allows the -1 step to erode successive values until the value of the current term is expressed as a single figure. At this point the base increase has no further effect and the value of successive terms decreases by 1. Another clue is given by why value each term achieves in comparison with its maximum in the current base. The "achievement" of successive terms diminishes.

The classic convergence proofs forGoodstein sequences use transfinite ordinals. We give a short introduction to ordinal numbers, with finite numbers defined in terms of finite sets, and show how this can be extended to introduce "transfinite" numbers $\omega,\ \omega+1, ...$ which are represented by infinite sets. We show how each weak Goodstein sequence is bounded above by a companion sequence whose initial terms are transfinite ordinals and whose terms decrease, thus, by a well known property of ordinals, inevitable arrive at zero.

Terms in the companion sequence are obtained by taking the expression of each term in the first sequence, written in terms of the current base, and replacing each occurrence of the base value by ω.

The same method is used with the strong Goodstein sequence, in which terms are written using hereditary base notation In this notation, the coefficients of a representation are also expressed in terms of the current base, greatly increasing the effect of a base increment and generating some of the largest numbers encountered in number theory. However we can still use the trick of producing a decreasing companion sequence of upper bounds. Here we limited ourselves to showing how one such term was generated, using an AI model to help us with the algebraic manipulations.